Level 5 Module 2
exemplar with comments

2015-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2015

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

Please stick the barcode label here.

Candidate Number									

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9 and 11.
2. This paper consists of TWO sections, A and B.
3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.
4. Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers must be exact.
7. No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

Answers written in the margins will not be marked.

1. Find $\frac{d}{dx}(x^5+4)$ from first principles. (4 marks)

$$\frac{d}{dx}(x^5+4)$$
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^5+4-(x^5+4)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^5-x^5}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{5x^4\Delta x+10x^3\Delta x^2+10x^2\Delta x^3+5x\Delta x^4+\Delta x^5}{\Delta x}$$
$$= 5x^4$$

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2. Let $y = x\sin x + \cos x$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

(b) Let k be a constant such that $x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + k\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0$ for all real values of x . Find the value of k .

(5 marks)

a. $\frac{dy}{dx} = \sin x + x\cos x - \sin x$

$= x\cos x$

$\frac{d^2y}{dx^2} = \cos x + (x)(-\sin x)$

$= \cos x - x\sin x$

b. $x\frac{d^2y}{dx^2} + k\frac{dy}{dx} + xy = 0$

$x(\cos x - x\sin x) + k(x\cos x) + x(x\sin x + \cos x) = 0$

$x\cos x - x^2\sin x + kx\cos x + x^2\sin x + x\cos x = 0$

$k = -2$

Answers written in the margins will not be marked.

3. (a) Find $\int \frac{1}{e^{2u}} \mathrm{d}u$.

(b) Using integration by substitution, evaluate $\int_1^9 \frac{1}{\sqrt{x}\, e^{2\sqrt{x}}} \mathrm{d}x$.

(7 marks)

a. $\int \frac{1}{e^{2u}} du$

$= \frac{1}{2} \int \frac{1}{(e^{2u})^2} d(e^{2u})$

$= \frac{1}{2}\left(-(e^{2u})^{-1}\right)$

$= -\frac{1}{2e^{2u}} + C$

b. let $u = \sqrt{x}$ $\quad du = \frac{1}{2\sqrt{x}} dx$

when $x = 9$, $u = 3$

When $x = 1$, $u = 1$

$\int_1^9 \frac{1}{\sqrt{x}\, e^{2\sqrt{x}}} dx$

$= \int_1^3 \frac{2}{e^{2u}} du$

$= \left[-\frac{1}{e^{2u}}\right]_1^3$

$= -\frac{1}{e^6} + \frac{1}{e^2}$

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4. (a) Using integration by parts, find $\int x^2 \ln x \, dx$.

(b) At any point (x, y) on the curve Γ , the slope of the tangent to Γ is $9x^2 \ln x$. It is given that Γ passes through the point $(1, 4)$. Find the equation of Γ .

(7 marks)

a. $\int x^2 \ln x \, dx$

$= \frac{1}{2} \int \ln x \, d(x^2)$

$= \frac{1}{2} [x^2 \ln x - \int x^2 d(\ln x)]$

$= \frac{1}{2} [x^2 \ln x - \int x \, dx]$

$= \frac{1}{2} (x^2 \ln x - \frac{x^2}{2})$

$= \frac{x^2}{2} (\ln x - \frac{1}{2}) + C_1$

b. $\int 9x^2 \ln x \, dx$

$= \frac{9x^2}{2} (\ln x - \frac{1}{2}) + C_2$

$4 = \frac{9}{2} (\ln 1 - \frac{1}{2}) + C_2$

$4 = -\frac{9}{4} + C_2$

$C_2 = \frac{25}{4}$

∴ The equation of Γ is :

$y = \frac{9x^2}{2} (\ln x - \frac{1}{2}) + \frac{25}{4}$

a. $\int x^2 \ln x \, dx$

$= \frac{1}{3} \int \ln x \, d(x^3)$

$= \frac{1}{3} [x^3 \ln x - \int x^3 d(\ln x)]$

$= \frac{1}{3} (x^3 \ln x - \frac{x^3}{3})$

$= \frac{x^3}{3} (\ln x - \frac{1}{3}) + C_1$

b. $\int 9x^2 \ln x \, dx$

$= 3x^3 (\ln x - \frac{1}{3}) + C_2$

$4 = 3(\ln 1 - \frac{1}{3}) + C_2$

$4 = -1 + C_2$

$C_2 = 5$

∴ The equation of Γ is :

$y = 3x^3 (\ln x - \frac{1}{3}) + 5$

Answers written in the margins will not be marked.

5. Solve the following systems of linear equations in real variables x, y, z:

(a) $\begin{cases} x + y + z = 2 \\ 2x + 3y - 3z = 4 \end{cases}$;

(b) $\begin{cases} x + y + z = 2 \\ 2x + 3y - 3z = 4 \\ 3x + 2y + kz = 6 \end{cases}$, where k is a real constant.

(6 marks)

a. $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let $z = t$, $y - 5t = 0$

$y = 5t$

$x + y + z = 0$

$x = -6t$

$\therefore (x, y, z) = (-6t, 5t, t)$

b. $3(-6t) + 2(5t) + k(t) = 6$

$(k-8)t = 6$

$t = \frac{6}{k-8}$

$\therefore (x, y, z) = \left(-\frac{36}{k-8}, \frac{30}{k-8}, \frac{6}{k-8}\right)$, when $k \neq 8$

There are no solutions when $k = 8$.

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6. (a) Let M be a 3×3 real matrix such that $M^T = -M$, where M^T is the transpose of M. Prove that $|M| = 0$.

(b) Let $A = \begin{pmatrix} -1 & a & b \\ -a & -1 & -8 \\ -b & 8 & -1 \end{pmatrix}$, where a and b are real numbers. Denote the 3×3 identity matrix by I.

(i) Using (a), or otherwise, prove that $|A+I| = 0$.

(ii) Someone claims that $A^3 + I$ is a singular matrix. Do you agree? Explain your answer.

(6 marks)

a. $M^T = -M$

$|M^T| = (-1)^3 |M|$

$|M| = -|M|$

$|M| = 0$

bi. $A + I = \begin{pmatrix} 0 & a & b \\ -a & 0 & -8 \\ -b & 8 & 0 \end{pmatrix}$

$(A+I)^T = \begin{pmatrix} 0 & -a & -b \\ a & 0 & 8 \\ b & -8 & 0 \end{pmatrix}$

$-(A+I) = \begin{pmatrix} 0 & -a & -b \\ a & 0 & 8 \\ b & -8 & 0 \end{pmatrix}$

$\because (A+I)^T = -(A+I)$

$\therefore |A+I| = 0$ (from (a))

ii $A^3 + I$

$= (A+I)(A^2 - AI + I^2)$

$\because |A+I| = 0$

$\therefore A^3 + I$ is a singular matrix.

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7. (a) Prove that $\sin^2 x \cos^2 x = \dfrac{1-\cos 4x}{8}$.

(b) Let $f(x) = \sin^4 x + \cos^4 x$.

(i) Express $f(x)$ in the form $A\cos Bx + C$, where A , B and C are constants.

(ii) Solve the equation $8f(x) = 7$, where $0 \le x \le \dfrac{\pi}{2}$.

(7 marks)

a. $\sin^2 x \cos^2 x$

$= \left(\dfrac{1-\cos 2x}{2}\right)\left(\dfrac{1+\cos 2x}{2}\right)$

$= \dfrac{1-\cos^2 2x}{4}$

$= \dfrac{1-\frac{1}{2}(\cos 4x+1)}{4}$

$= \dfrac{1-\cos 4x}{8}$

bi. $f(x) = \sin^4 x + \cos^4 x$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$= \left(\dfrac{1-\cos 2x}{2} + \dfrac{1+\cos 2x}{2}\right) - \dfrac{1-\cos 4x}{4}$

$= 1 - \dfrac{1}{4} + \dfrac{\cos 4x}{4}$

$= \dfrac{1}{4}\cos 4x + \dfrac{3}{4}$

ii. $8f(x) = 7$ $\qquad 0 \le x \le \frac{\pi}{2}$

$2\cos 4x + 6 = 7$ $\qquad 0 \le 4x \le 2\pi$

$\cos 4x = \dfrac{1}{2}$

$4x = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$

$x = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$

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8. (a) Using mathematical induction, prove that $\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx = \sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=1}^{567}\cos\frac{k\pi}{7}$.

(8 marks)

a. Let $P(n)$ be $\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx = \sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$

For $P(1)$: $LHS = \sin\frac{x}{2}\sum_{k=1}^{1}\cos kx$

$= \sin\frac{x}{2}\cos x$

$RHS = \sin\frac{x}{2}\cos\frac{2x}{2}$

$= \sin\frac{x}{2}\cos x$

$\therefore LHS = RHS$

$\therefore P(1)$ is true.

Assume that $P(t)$ is true, where t is a positive integer.

i.e. $\sin\frac{x}{2}\sum_{k=1}^{t}\cos kx = \sin\frac{tx}{2}\cos\frac{(t+1)x}{2}$

$P(t+1) = \sin\frac{x}{2}\sum_{k=1}^{t+1}\cos kx$

$= \sin\frac{x}{2}\left(\sum_{k=1}^{t}\cos kx + \cos(t+1)x\right)$

$= \sin\frac{tx}{2}\cos\frac{(t+1)x}{2} + \sin\frac{x}{2}\cos(t+1)x$

$= \sin\frac{(t+1)x}{2}\cos\frac{(t+2)x}{2}$

$\therefore P(t+1)$ is also true.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Answers written in the margins will not be marked.

b. $\sum_{k=1}^{567} \cos\frac{k\pi}{7} = \frac{\sin\frac{567\pi}{7}\cos\frac{568\pi}{7}}{\sin\frac{\pi}{7}}$

$= \frac{-1\times 1}{1}$

$= -1$

$$\sum_{k=1}^{567}\cos\frac{k\pi}{7} = \frac{\sin\frac{567\times\frac{\pi}{7}}{2}\cos\frac{568\times\frac{\pi}{7}}{2}}{\sin\frac{\pi}{14}}$$

$$= \frac{\sin\frac{81\pi}{2}\cos\frac{284\pi}{7}}{\sin\frac{\pi}{14}}$$

$$= -1$$

SECTION B (50 marks)

9. Define $f(x)=\dfrac{x^2+12}{x-2}$ for all $x \neq 2$.

(a) Find $f'(x)$. (2 marks)

(b) Prove that the maximum value and the minimum value of $f(x)$ are -4 and 12 respectively. (4 marks)

(c) Find the asymptote(s) of the graph of $y=f(x)$. (3 marks)

(d) Find the area of the region bounded by the graph of $y=f(x)$ and the horizontal line $y=14$. (4 marks)

a. $f'(x)=\dfrac{2x(x-2)-(x^2+12)}{(x-2)^2}$

$=\dfrac{x^2-4x-12}{(x-2)^2}$

$=\dfrac{(x-6)(x+2)}{(x-2)^2}$

b. $f'(x)=0$ when $x=-2$ or 6

x	$x<-2$	$x=-2$	$-2<x<6$	$x=6$	$x>6$
$f(x)$	$\nearrow$	-4	$\searrow$	12	$\nearrow$
$f'(x)$	$+$	0	$-$	0	$+$

$\therefore$ The maximum value is -4 and minimum value is 12.

c. $\lim\limits_{x\to\infty}\dfrac{x^2+12}{x-2}$ does not exist.

$\therefore$ There are no horizontal asyntote.

$\lim\limits_{x\to 2^+}=+\infty$

$\lim\limits_{x\to 2^-}=-\infty$

$\therefore$ $x=2$ is the vertical asymtote of $y=f(x)$

$\lim\limits_{x\to\infty}\dfrac{x^2+12}{x(x-2)}$

$=\lim\limits_{x\to\infty}\dfrac{1+\frac{12}{x^2}}{1+\frac{-2}{x}}$

$=1$

Answers written in the margins will not be marked.

$\lim_{x\to\infty} \frac{x^2+12}{x-2} - x$

$= \lim_{x\to\infty} \frac{x^2+12-(x^2-2x)}{x-2}$

$= \lim_{x\to\infty} \frac{2x+12}{x-2}$

$= \lim_{x\to\infty} \frac{2+\frac{12}{x}}{1-\frac{2}{x}}$

$= 2$

$\therefore$ The oblique asyntote is $y=x+2$.

d. For $y=14$, $\frac{x^2+12}{x-2}=14$

$x^2+12=14x-28$

$(x-4)(x-10)=0$

$x=4$ or 10

Area $= \int_4^{10} (14-\frac{x^2+12}{x-2})dx$ Let $u=x-2$

$\cancel{= [14x]_4^{10} - \int_4^{10} \frac{x^2+12}{x-2}dx}$ $du=dx$

$\cancel{= 84 -}$ When $x=10$, $u=8$

$= \int_2^8 (14-\frac{u^2+4u+16}{u})du$ When $x=4$, $u=2$

$= [14u]_2^8 - [\frac{u^2}{2}]_2^8 - [4u]_2^8 - 16[\ln|u|]_2^8$

$= 60-30-16(\ln 8 - \ln 2)$

$= 30-16(2\ln 2)$

$= 30-32\ln 2$ //

10. OAB is a triangle. P is the mid-point of OA. Q is a point lying on AB such that $AQ:QB=1:2$ while R is a point lying on OB such that $OR:RB=3:1$. PR and OQ intersect at C.

(a) (i) Let t be a constant such that $PC:CR=t:(1-t)$.
By expressing $\overrightarrow{OQ}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, find the value of t.

(ii) Find $CQ:OQ$.

(7 marks)

(b) Suppose that $\overrightarrow{OA}=20\mathbf{i}-6\mathbf{j}-12\mathbf{k}$, $\overrightarrow{OB}=16\mathbf{i}-16\mathbf{j}$ and $\overrightarrow{OD}=\mathbf{i}+3\mathbf{j}-6\mathbf{k}$, where O is the origin. Find

(i) the area of ΔOAB,

(ii) the volume of the tetrahedron $ABCD$.

(5 marks)

ai. $\overrightarrow{OQ}=\frac{2\overrightarrow{OA}+\overrightarrow{OB}}{3}=\frac{2}{3}\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}$

$\because$ OC and OQ are collinear

$\therefore \overrightarrow{OC}=k\overrightarrow{OQ}$

$\overrightarrow{OC}=\frac{1-t}{2}\overrightarrow{OA}+\frac{3t}{4}\overrightarrow{OB}$

$\frac{\frac{2}{3}}{\frac{1-t}{2}}=\frac{\frac{1}{3}}{\frac{3t}{4}}$

$\frac{t}{2}=\frac{1-t}{6}$

$6t=2-2t$

$t=\frac{1}{4}$

ii $\overrightarrow{OC}=\frac{1-\frac{1}{4}}{2}\overrightarrow{OA}+\frac{3\times\frac{1}{4}}{4}\overrightarrow{OB}$

$=\frac{3}{8}\overrightarrow{OA}+\frac{3}{16}\overrightarrow{OB}$

$\overrightarrow{CQ}=\overrightarrow{OQ}-\overrightarrow{OC}$

$=(\frac{2}{3}-\frac{3}{8})\overrightarrow{OA}+(\frac{1}{3}-\frac{3}{16})\overrightarrow{OB}$

$=\frac{7}{24}\overrightarrow{OA}+\frac{7}{48}\overrightarrow{OB}$

$CQ:OQ=\frac{7}{24}:\frac{2}{3}$

$=21:48$

$=7:16$

b) i. Area of $\Delta OAB = |\vec{OA} \times \vec{OB}|$

$$= \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 20 & -6 & -12 \\ 16 & -16 & 0 \end{matrix} \right\|$$

$$= |-192\vec{i} - 192\vec{j} - 224\vec{k}|$$

$$= 352$$

ii. Volume $= (OA \times OB) \cdot \vec{OD} \times \frac{1}{6}$

$$= \begin{vmatrix} 20 & -6 & -12 \\ 16 & -16 & 0 \\ 1 & 3 & -6 \end{vmatrix} \times \frac{1}{6}$$

$$= 576 \times \frac{1}{6}$$

$$= 96$$

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11. (a) Let λ and μ be real numbers such that $\mu - \lambda \neq 2$. Denote the 2×2 identity matrix by I.

Define $A = \dfrac{1}{\lambda - \mu + 2}(I - \mu I + M)$ and $B = \dfrac{1}{\lambda - \mu + 2}(I + \lambda I - M)$, where $M = \begin{pmatrix} \lambda & 1 \\ \lambda - \mu + 1 & \mu \end{pmatrix}$.

(i) Evaluate AB , BA and $A + B$.

(ii) Prove that $A^2 = A$ and $B^2 = B$.

(iii) Prove that $M^n = (\lambda + 1)^n A + (\mu - 1)^n B$ for all positive integers n .

(8 marks)

(b) Using (a), or otherwise, evaluate $\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315}$. (4 marks)

a i. $A = \frac{1}{\lambda-\mu+2}(I-\mu I+M)$

$= \frac{1}{\lambda-\mu+2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}\mu&0\\0&\mu\end{pmatrix}+\begin{pmatrix}\lambda&1\\\lambda-\mu+1&\mu\end{pmatrix}\right)$

$= \frac{1}{\lambda-\mu+2}\begin{pmatrix}1-\mu+\lambda&1\\1-\mu+\lambda&1\end{pmatrix}$

$B = \frac{1}{\lambda-\mu+2}(I+\lambda I-M)$

$= \frac{1}{\lambda-\mu+2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}-\begin{pmatrix}\lambda&1\\\lambda-\mu+1&\mu\end{pmatrix}\right)$

$= \frac{1}{\lambda-\mu+2}\begin{pmatrix}1&-1\\\mu-\lambda-1&1+\lambda-\mu\end{pmatrix}$

$AB = \frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}1-\mu+\lambda&1\\1-\mu+\lambda&1\end{pmatrix}\begin{pmatrix}1&-1\\\mu-\lambda-1&1+\lambda-\mu\end{pmatrix}$

$= \frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$= 0$

$BA = \frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}1&-1\\\mu-\lambda-1&1+\lambda-\mu\end{pmatrix}\begin{pmatrix}1-\mu+\lambda&1\\1-\mu+\lambda&1\end{pmatrix}$

$= \frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$= 0$

Answers written in the margins will not be marked.

$$A+B=\frac{1}{\lambda-\mu+2}\left[\begin{pmatrix}1-\mu+\lambda & 1\\ 1-\mu+\lambda & 1\end{pmatrix}+\begin{pmatrix}1 & -1\\ \mu-\lambda-1 & 1+\lambda-\mu\end{pmatrix}\right]$$

$$=\frac{1}{\lambda-\mu+2}\begin{pmatrix}2-\mu+\lambda & 0\\ 0 & 2-\mu+\lambda\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$=I$$

ii $(A+B)^2=A^2+B^2+BA+AB$

$I=A^2+B^2$

$I=A+B$

$\therefore A^2=A$ and $B^2=B$

iii

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b. $\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315}$

$= \left[2\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\right]^{315}$

let $\lambda = 2$ and $\mu = 3$

$M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

$M^{315} = (3)^{315}A + (2)^{315}B$ (From (a))

$= (3)^{315}\left[\frac{1}{1}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right] + 2^{315}\left[1\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\right]$

$= 3^{315}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 2^{315}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315} = 2^{315}\cdot 3^{315}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 2^{315}\cdot 2^{315}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 6^{315} \\ 0 & 6^{315} \end{pmatrix} + \begin{pmatrix} 2^{630} & -2^{630} \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2^{630} & 6^{315} - 2^{630} \\ 0 & 6^{315} \end{pmatrix}$ //

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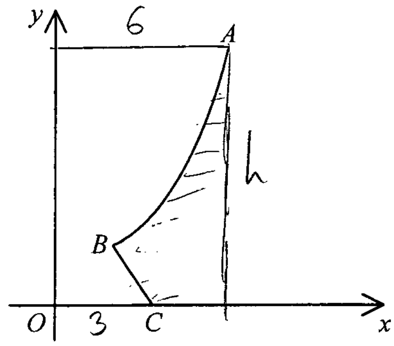
12. (a) In the figure, the curve Γ consists of the curve AB, the line segments BC and CO, where O is the origin, B lies in the first quadrant and C lies on the x-axis. The equations of AB and BC are $x^2-4y+8=0$ and $3x+y-9=0$ respectively.



(i) Find the coordinates of B.

(ii) Let h be the y-coordinate of A, where $h>3$. A cup is formed by revolving Γ about the y-axis. Prove that the capacity of the cup is $\pi(2h^2-8h+25)$.

(7 marks)

(b) A cup described in (a)(ii) is placed on a horizontal table. The radii of the base and the lip of the cup are 3 cm and 6 cm respectively.

(i) Find the capacity of the cup.

(ii) Water is poured into the cup at a constant rate of 24π cm^3/s. Find the rate of change of the depth of water when the volume of water in the cup is 35π cm^3.

(6 marks)

ai. $\begin{cases} x^2-4y+8=0 & —(1) \\ 3x+y-9=0 & —(2) \end{cases}$

From (2), $y=9-3x$ —(3)

Sub (3) into (1), $x^2-4(9-3x)+8=0$

$x^2+12x-28=0$

$(x-2)(x+14)=0$

$x=2$ or -14

$\because$ B is in the 1st Quad.

$\therefore$ $x=2$

When $x=2$, $y=9-6$

$=3$

$\therefore$ B(2, 3).

Answers written in the margins will not be marked.

ii Capacity $= \pi\int_3^h(\sqrt{4y-8})^2 dy + \pi\int_0^3(\frac{9-y}{3})^2 dy$

$= \pi\int_3^h(4y-8)dy + \pi\int_0^3(\frac{81-18y+y^2}{9})dy$

$= \pi[2y^2-8y]_3^h + \pi[9y-y^2+\frac{y^3}{27}]_0^3$

$= \pi(2h^2-8h+6)+\pi(19-0)$

$= \pi(2h^2-8h+25)$

bi. $6^2 - 4h + 8 = 0$

$h = 11$

Capacity $= \pi(2\times11^2-8\times11+25)$

$= 179\pi$

ii. Let V be the volume of water.

$\frac{dV}{dt} = \pi(4h-8)\frac{dh}{dt}$

When volume is 35π,

$2h^2-8h+25=35$

$h^2-4h-5=0$

$h=-1$ (rejected) or 5

$\frac{dV}{dt} = 24\pi$

$24\pi = \pi(4\times5-8)\frac{dh}{dt}$

$\frac{dh}{dt} = 2$ cm/s

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END OF PAPER

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Comments

The candidate demonstrates comprehensive knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by applying them successfully at a sophisticated level to a wide range of unfamiliar situations as in Questions 9, 10, 11 and 12.

He/She is able to communicate and express views and arguments precisely and logically using mathematical language, notations and diagrams, such as in using limit notations in Question 1, integration symbols in Questions 3 and 4, matrix notations in Question 6 and a derivative test table in Question 9(b).

He/She also provides complex mathematical proofs in a logical, rigorous and concise manner in Questions 7(a), 9(b) and 12(a).

It can be concluded that the candidate has the ability to integrate knowledge and skills from different areas of the curriculum in handling complex tasks using a variety of strategies.