Level 5 Module 2
exemplar with comments

2017-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2017

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8.30 am – 11.00 am (2½ hours)

This paper must be answered in English

Please stick the barcode label here.

Candidate Number									

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

****

SECTION A (50 marks)

Answers written in the margins will not be marked.

1. Find $\frac{d}{d\theta}\sec 6\theta$ from first principles. (5 marks)

$$\frac{d}{d\theta}\sec 6\theta = \lim_{\Delta\theta \to 0}\frac{\sec 6(\Delta\theta+\theta) - \sec 6\theta}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0}\frac{\frac{1}{\cos(6\Delta\theta+6\theta)} - \frac{1}{\cos 6\theta}}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0}\frac{\frac{\cos 6\theta - \cos(6\Delta\theta+6\theta)}{\cos(6\Delta\theta+6\theta)\cos 6\theta}}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0}\frac{\frac{-2\sin(6\theta+3\Delta\theta)\sin(-3\Delta\theta)}{\cos(6\Delta\theta+6\theta)\cos 6\theta}}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0}\frac{2\sin(6\theta+3\Delta\theta)}{\cos(6\Delta\theta+6\theta)\cos 6\theta}\cdot\frac{\sin(3\Delta\theta)}{3\Delta\theta}\cdot 3$$

$$= \frac{6\sin 6\theta}{\cos^2 6\theta}$$

$$= 6\sec 6\theta\tan 6\theta$$

Answers written in the margins will not be marked.

2. Let $(1+ax)^8 = \sum_{k=0}^{8} \lambda_k x^k$ and $(b+x)^9 = \sum_{k=0}^{9} \mu_k x^k$, where a and b are constants. It is given that $\lambda_2 : \mu_7 = 7:4$ and $\lambda_1 + \mu_8 + 6 = 0$. Find a. (5 marks)

$(r+1)$th term of $(1+ax)^8 = C^8_r a^r x^r$

$(r+1)$th term of $(b+x)^9 = C^9_r b^{9-r} x^r$

$$\frac{\lambda_2}{\mu_7} = \frac{7}{4}$$

$$\frac{C^8_2 a^2}{C^9_7 b^{9-7}} = \frac{7}{4}$$

$$\frac{28a^2}{36b^2} = \frac{7}{4}$$

$$\frac{a^2}{b^2} = \frac{9}{4}$$

$$\frac{a}{b} = \pm\frac{3}{2} \Rightarrow b = \pm\frac{2}{3}a$$

$\lambda_1 + \mu_8 + 6 = 0$

$C^8_1 a + C^9_8 b^{9-8} + 6 = 0$

$8a + 9b + 6 = 0$

$8a + 6a + 6 = 0$ or $8a - 6a + 6 = 0$

$a = -\frac{3}{7}$ or $a = -3$

$\therefore$ when $b = -\frac{2}{7}$, $a = -\frac{3}{7}$

when $b = 2$, $a = -3$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

3. P is a point lying on AB such that $AP:PB=3:2$. Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$, where O is the origin.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $|\mathbf{a}|=45$, $|\mathbf{b}|=20$ and $\cos\angle AOB=\frac{1}{4}$. Find

(i) $\mathbf{a}\cdot\mathbf{b}$,

(ii) $|\overrightarrow{OP}|$.

(5 marks)

(a) $\overrightarrow{OP} = \dfrac{2\overrightarrow{OA}+3\overrightarrow{OB}}{5}$

$= \dfrac{2\vec{a}+3\vec{b}}{5}$

$= \frac{2}{5}\vec{a}+\frac{3}{5}\vec{b}$

(b)(i) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\angle AOB$

$= 45\cdot 20\cdot \frac{1}{4}$

$= 225$

(ii) $\overrightarrow{OP} = \frac{2}{5}\vec{a}+\frac{3}{5}\vec{b}$

$|\overrightarrow{OP}|^2 = (\frac{2}{5}\vec{a}+\frac{3}{5}\vec{b})\cdot(\frac{2}{5}\vec{a}+\frac{3}{5}\vec{b})$

$= \frac{4}{25}|\vec{a}|^2+\frac{12}{25}\vec{a}\cdot\vec{b}+\frac{9}{25}|\vec{b}|^2$

$= \frac{4}{25}(45)^2+\frac{12}{25}(225)+\frac{9}{25}(20)^2$

$= 576$

$\therefore |\overrightarrow{OP}| = \sqrt{576}$

$= 24$

Answers written in the margins will not be marked.

4. (a) Using integration by parts, find $\int x^2 e^{-x}\,dx$.

(b) Find the area of the region bounded by the graph of $y = x^2 e^{-x}$, the x-axis and the straight line $x = 6$.

(6 marks)

(a) $\int x^2 e^{-x} dx = -\int x^2 \, d(e^{-x})$

$= -x^2 e^{-x} + \int e^{-x} d(x^2)$

$= -x^2 e^{-x} + 2\int x e^{-x} dx$

$= -x^2 e^{-x} - 2\int x \, d(e^{-x})$

$= -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x} dx$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

(b) area $= \int_0^6 x^2 e^{-x} dx$

$= \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^6$

$= -36e^{-6} - 12e^{-6} - 2e^{-6} - (-2)$

$= -50e^{-6} + 2$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

5. Consider the following system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + 2y - z = 11 \\ 3x + 8y - 11z = 49 \\ 2x + 3y + hz = k \end{cases}, \text{ where } h, k \in \mathbf{R} .$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of h.

(ii) Express z in terms of h and k.

(b) Assume that (E) has infinitely many solutions. Solve (E).

(6 marks)

(a)(i) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & -11 \\ 2 & 3 & h \end{vmatrix} \neq 0$

$8h - 44 - 9 - (-32 - 33 + 6h) \neq 0$

$2h + 12 \neq 0$

$2h \neq -12$

$\therefore h \neq -6$

$\therefore$ range is $h < -6$ or $h > -6$

(ii) $z = \dfrac{\begin{vmatrix} 1 & 2 & 11 \\ 3 & 8 & 49 \\ 2 & 3 & k \end{vmatrix}}{2h+12}$

$= \dfrac{8k + 196 + 99 - 176 - 147 - 6k}{2h+12}$

$= \dfrac{2k - 28}{2h + 12}$

$= \dfrac{k - 14}{h + 6}$

(b) h will be -6 when (E) has infinitely many solutions

$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & -6 & k \end{array}\right)$

$\begin{array}{l} R_2 - 3R_1 = R_2 \\ \sim \\ R_3 - 2R_1 = R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 0 & 2 & -8 & 16 \\ 0 & -1 & -4 & k-22 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & -1 & -4 & k-22 \end{array}\right)$

Answers written in the margins will not be marked.

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & -8 & k-14 \end{array} \right)$$

$\therefore -8z = k-14$

$z = -\frac{1}{8}k + \frac{7}{4}$

$y - 4(-\frac{1}{8}k + \frac{7}{4}) = 8$

$y + \frac{1}{2}k - 7 = 8$

$y = 15 - \frac{1}{2}k$

$x + 2(15 - \frac{1}{2}k) - (-\frac{1}{8}k + \frac{7}{4}) = 11$

$x + 30 - k + \frac{1}{8}k - \frac{7}{4} = 11$

$x = \frac{7}{8}k - \frac{69}{4}$

$$\therefore \begin{cases} x = \frac{7}{8}k - \frac{69}{4} \\ y = -\frac{1}{2}k + 15 \\ z = -\frac{1}{8}k + \frac{7}{4} \end{cases}$$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

6. A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.

(a) Let A cm^2 be the wet curved surface area of the container and h cm be the depth of water in the container. Prove that $A = \frac{15}{16}\pi h^2$.

$\frac{dh}{dt}$

(b) The depth of water in the container increases at a constant rate of $\frac{3}{\pi}$ cm/s . Find the rate of change of the wet curved surface area of the container when the volume of water in the container is 96π cm^3.

$\frac{dA}{dt}$

(7 marks)

(a) let r be the radius of water surface

$$\frac{h}{20} = \frac{r}{15}$$

$$r = \frac{3}{4}h$$

$$A = \pi r\left(\sqrt{h^2+r^2}\right)$$

$$= \pi\left(\frac{3}{4}h\right)\sqrt{h^2+\left(\frac{3}{4}h\right)^2}$$

$$= \pi\left(\frac{3}{4}h\right)\sqrt{h^2+\frac{9}{16}h^2}$$

$$= \frac{3}{4}\pi h\sqrt{\frac{25}{16}h^2}$$

$$= \frac{3}{4}\pi h\left(\frac{5}{4}h\right)$$

$$= \frac{15}{16}\pi h^2$$

(b)

$$96\pi = \frac{1}{3}\pi r^2 h$$

$$96\pi = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h$$

$$h = 8$$

$$A = \frac{15}{16}\pi h^2$$

$$\frac{dA}{dt} = \frac{15}{16}\pi(2h)\frac{dh}{dt}$$

$$= \frac{15}{16}\pi(2\times 8)\left(\frac{3}{\pi}\right)$$

$$= 45 \text{ cm}^2/\text{s}$$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

7. (a) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

45° 90°

(b) Let $\frac{\pi}{4} < x < \frac{\pi}{2}$.

(i) Prove that $\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)} = \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$.

(ii) Solve the equation $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$.

(8 marks)

(a) L.H.S. $= \sin 3x$

$= \sin(x+2x)$

$= \sin x \cos 2x + \cos x \sin 2x$

$= \sin x (1-2\sin^2 x) + 2\sin x \cos^2 x$

$= \sin x - 2\sin^3 x + 2\sin x(1-\sin^2 x)$

$= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$=$ R.H.S.

$\therefore \sin 3x = 3\sin x - 4\sin^3 x$

(b)(i) L.H.S. $= \dfrac{\sin 3(x-\frac{\pi}{4})}{\sin(x-\frac{\pi}{4})}$

$= \dfrac{3\sin(x-\frac{\pi}{4}) - 4\sin^3(x-\frac{\pi}{4})}{\sin(x-\frac{\pi}{4})}$

$= 3 - 4\sin^2(x-\frac{\pi}{4})$

$= 3 - 4[\sin x \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos x]^2$

$= 3 - 4(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x)^2$

$= 3 - 4(\frac{1}{2}\sin^2 x - \sin x\cos x + \frac{1}{2}\cos^2 x)$

$= 1 - \sin x \cos x$

R.H.S. $= \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$

$=$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

(b)(ii) $\frac{\cos 3x+\sin 3x}{\cos x-\sin x}=2$

$\frac{\sin 3(x-\frac{\pi}{4})}{\sin(x-\frac{\pi}{4})}=2$

$\frac{\sin(x-\frac{\pi}{4})\cos 2(x-\frac{\pi}{4})+\sin 2(x-\frac{\pi}{4})\cos(x-\frac{\pi}{4})}{\sin(x-\frac{\pi}{4})}=2$

$\cos 2(x-\frac{\pi}{4})+\frac{2\cancel{\sin(x-\frac{\pi}{4})}\cos^2(x-\frac{\pi}{4})}{\cancel{\sin(x-\frac{\pi}{4})}}=2$

$2\cos^2(x-\frac{\pi}{4})+2\cos^2(x-\frac{\pi}{4})-1-2=0$

$\cos(x-\frac{\pi}{4})=\pm\sqrt{\frac{3}{4}}$

$x-\frac{\pi}{4}=\frac{\pi}{6}$ or $\frac{11\pi}{6}$ (rej) or $\frac{5\pi}{6}$ (rej) or $\frac{7\pi}{6}$ (rej)

$\therefore x=\frac{5\pi}{12}$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

8. Let $f(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $P(e^3, 7)$ and $f'(x) = \frac{1}{x}\ln x^2$ for all $x > 0$. Find

(a) the equation of the tangent to Γ at P,

(b) the equation of Γ,

(c) the point(s) of inflexion of Γ.

(8 marks)

(a) $f'(e^3) = \frac{1}{e^3}\ln(e^3)^2$

$= \frac{6}{e^3}$

$\therefore$ equation: $y - 7 = \frac{6}{e^3}(x - e^3)$

$y - 7 = \frac{6}{e^3}x - 6$

$y = \frac{6}{e^3}x + 1$

(b) $f'(x) = \frac{1}{x}\ln x^2$

$f(x) = \int \frac{1}{x}\ln x^2\,dx$

$= 2\int x^{-1}\ln x\,dx$

$= 2\int \ln x\,d(\ln x)$

$= (\ln x)^2 + C$

Sub $(e^3, 7)$, $7 = (\ln e^3)^2 + C$

$C = -2$

$\therefore$ equation of Γ is $y = (\ln x)^2 - 2$

(c) $f'(x) = \frac{1}{x}\ln x^2 = 2x^{-1}\ln x$

$f''(x) = 2x^{-1}\frac{1}{x} + 2(-1)(x^{-2})\ln x$

$= 2x^{-2} - 2x^{-2}\ln x$

$= 2x^{-2}(1 - \ln x)$

When $f''(x) = 0$, $\frac{2(1-\ln x)}{x^2} = 0$

$x = e$

x	$x < e$	$x = e$	$x > e$
$f''(x)$	+	0	−

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

When $x=e$, $y=-1$

$\therefore$ pt of inflexion is $(e,-1)$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

SECTION B (50 marks)

9. Define $f(x) = \dfrac{x^2 - 5x}{x+4}$ for all $x \neq -4$. Denote the graph of $y = f(x)$ by G.

   (a) Find the asymptote(s) of G. (3 marks)

   (b) Find $f'(x)$. (2 marks)

   (c) Find the maximum point(s) and the minimum point(s) of G. (4 marks)

   (d) Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis. (4 marks)

(a) vertical asymptote is $x = -4$

$$\begin{array}{r} x-9 \\ x+4 \overline{)\, x^2 - 5x} \\ \underline{x^2+4x} \\ -9x \\ \underline{-9x-36} \\ 36 \end{array}$$

$\therefore \dfrac{x^2-5x}{x+4} = x - 9 + \dfrac{36}{x+4}$

$\therefore$ oblique asymptote is $y = x - 9$

(b) $f(x) = (x^2-5x)(x+4)^{-1}$

$f'(x) = (x^2-5x)(-1)(x+4)^{-2} + (x+4)^{-1}(2x-5)$

$= (x+4)^{-2}[(x^2-5x)(-1) + (x+4)(2x-5)]$

$= \dfrac{x^2+8x-20}{(x+4)^2}$

$= \dfrac{(x-2)(x+10)}{(x+4)^2}$

(c) When $f'(x) = 0$, $\dfrac{(x-2)(x+10)}{(x+4)^2} = 0$

$x = 2$ or -10

x	$x<-10$	$x=-10$	$-10<x<-4$	$x=-4$	$-4<x<2$	$x=2$	$x>2$
$f'(x)$	+	0	−	undefined	−	0	+

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

When $x=-10$, $y=-25$

When $x=2$, $y=-1$

$\therefore$ maximum point is $(-10,-25)$

minimum point is $(2,-1)$

(d) When $y=0$, $\dfrac{x^2-5x}{x+4}=0$

$$x^2-5x=0$$

$$x=5 \text{ or } 0$$

$$\text{volume} = \pi\int_0^5 \left(\frac{x^2-5x}{x+4}\right)^2 dx$$

$$= \pi\int_0^5 \left(\frac{x^4-10x^3+25x^2}{x^2+8x+16}\right) dx$$

$$= \pi\int_0^5 \frac{x^2(x^2-10x+25)}{x^2+4x+16} dx$$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

10. ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE:EC=1:r$. AB produced and DE produced meet at the point F. It is given that $DE:EF=1:10$. Let $\overrightarrow{OA}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, $\overrightarrow{OB}=4\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\overrightarrow{OC}=8\mathbf{i}-3\mathbf{j}-2\mathbf{k}$, where O is the origin.

(a) By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r, find r. (4 marks)

(b) (i) Find $\overrightarrow{AD}\cdot\overrightarrow{DE}$.

(ii) Are B, D, C and F concyclic? Explain your answer. (5 marks)

(c) Let $\overrightarrow{OP}=3\mathbf{i}+10\mathbf{j}-4\mathbf{k}$. Denote the circumcentre of ΔBCF by Q. Find the volume of the tetrahedron $ABPQ$. (3 marks)

(a) $\overrightarrow{AE}=\overrightarrow{AB}+\overrightarrow{BE}$

$=\overrightarrow{OB}-\overrightarrow{OA}+\frac{1}{1+r}\overrightarrow{BC}$

$=\overrightarrow{OB}-\overrightarrow{OA}+\frac{1}{1+r}\overrightarrow{OC}-\frac{1}{1+r}\overrightarrow{OB}$

$=4\vec{i}+4\vec{j}-\vec{k}-2\vec{i}-3\vec{j}+2\vec{k}+\frac{8}{1+r}\vec{i}-\frac{3}{1+r}\vec{j}-\frac{2}{1+r}\vec{k}$
$-\frac{4}{1+r}\vec{i}-\frac{4}{1+r}\vec{j}+\frac{1}{1+r}\vec{k}$

$=(2+\frac{4}{1+r})\vec{i}+(1-\frac{7}{1+r})\vec{j}+(1-\frac{1}{1+r})\vec{k}$

$\overrightarrow{AF}=\overrightarrow{AE}+\overrightarrow{EF}$

$=\overrightarrow{AE}+\frac{1}{10}\overrightarrow{ED}$

$=\overrightarrow{AE}+\frac{1}{10}(\overrightarrow{EC}+\overrightarrow{CD})$

$=\overrightarrow{AE}+\frac{1}{10}(\frac{r}{1+r}\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CA})$

$=\overrightarrow{AE}+\frac{4r}{10(1+r)}\vec{i}-\frac{7}{10(1+r)}\vec{j}-\frac{r}{10(1+r)}\vec{k}-\frac{3}{10}\vec{i}+\frac{3}{10}\vec{j}$

$=(\frac{17}{10}+\frac{4}{1+r}+\frac{4r}{10(1+r)})\vec{i}+(\frac{13}{10}-\frac{7}{1+r}-\frac{7}{10(1+r)})\vec{j}+(1-\frac{1}{1+r}-\frac{r}{10(1+r)})\vec{k}$

$\cancel{\overrightarrow{EF}=\overrightarrow{EA}+\overrightarrow{AF}=\overrightarrow{AF}-\overrightarrow{AE}}$

$\cancel{(\frac{3}{10}+\frac{4r}{10(1+r)})\vec{i}+(\frac{3}{10}-\frac{10}{10(1+r)})\vec{j}-\frac{r}{10(1+r)}\vec{k}}$

$\frac{\overrightarrow{AF}+9\overrightarrow{AE}}{10}=\overrightarrow{AD}$

$\overrightarrow{AF}+9\overrightarrow{AE}=10\overrightarrow{AD}=5\overrightarrow{AC}$

$\therefore\ \frac{17}{10}+\frac{4}{1+r}+\frac{4r}{10(1+r)}+18+\frac{36}{1+r}=30$

$\frac{40+4r+360}{\cancel{10}(1+r)}=\frac{103}{\cancel{10}}$

$40+4r+360=103+103r$

$r=3$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

$(b)(i)$ $\overrightarrow{AD}\cdot\overrightarrow{DE} = \overrightarrow{AD}\cdot(\overrightarrow{DC}+\overrightarrow{CE})$

$= \frac{1}{3}\overrightarrow{AC}\cdot(\frac{1}{2}\overrightarrow{AC}+\frac{3}{4}\overrightarrow{CB})$

$= (3\vec{i}-3\vec{j})\cdot(\frac{9}{4}\vec{j}+\frac{3}{4}\vec{k})$

$= -\frac{27}{4}$

(ii)

(c) volume $= |AB\cdot(AP\times AQ)|$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

11. (a) Using $\tan^{-1}\sqrt{2}-\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)=\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$, evaluate $\int_0^1 \frac{1}{x^2+2x+3}\,dx$. (3 marks)

45°

(b) (i) Let $0 \le \theta \le \frac{\pi}{4}$. Prove that $\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$ and $\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$.

(ii) Using the substitution $t=\tan\theta$, evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$.

(5 marks)

(c) Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta=\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$. (2 marks)

(d) Evaluate $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,d\theta$. (3 marks)

(a) $\int_0^1 \frac{1}{x^2+2x+3}\,dx$

$=\int_0^1 \frac{1}{(x^2+2x+1^2-1^2)+3}\,dx$

$=\int_0^1 \frac{1}{(x+1)^2+2}\,dx$

let $x+1=\sqrt{2}\tan\theta$

$dx=\sqrt{2}\sec^2\theta\,d\theta$

when $x=1$, $\theta=\tan^{-1}\sqrt{2}$

when $x=0$, $\theta=\tan^{-1}\frac{\sqrt{2}}{2}$

$\therefore \int_0^1 \frac{1}{(x+1)^2+2}\,dx=\int_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}} \frac{\sqrt{2}\cancel{\sec^2\theta}}{2\cancel{\sec^2\theta}}\,d\theta$

$=\frac{\sqrt{2}}{2}\left[\theta\right]_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}}$

$=\frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2}-\tan^{-1}\frac{\sqrt{2}}{2}\right)$

$=\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(b)(i) $\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2\tan\theta}{\sec^2\theta}=\frac{2\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}=2\sin\theta\cos\theta=\sin 2\theta$

$\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\tan^2\theta}{\sec^2\theta}=\cos^2\theta-\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}=\cos^2\theta-\sin^2\theta=\cos 2\theta$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

(b)(ii) $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\frac{2\tan\theta}{1+\tan^2\theta}+\frac{1-\tan^2\theta}{1+\tan^2\theta}+2}\, d\theta$

Sub $t=\tan\theta$

$dt = \sec^2\theta\, d\theta$

When $\theta=\frac{\pi}{4}$, $t=1$

When $\theta=0$, $t=0$

$\int_0^1 \frac{\frac{1}{t^2+1}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+2}\, dt$

$= \int_0^1 \frac{1+t^2}{\frac{2t+1-t^2+2+2t^2}{1+t^2}}\, dt$

$= \int_0^1 \frac{1}{t^2+2t+3}\, dt$

$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(c) let $\theta=\frac{\pi}{4}-u$

$d\theta=-du$

When $\theta=\frac{\pi}{4}$, $u=0$

When $\theta=0$, $u=\frac{\pi}{4}$

$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\, d\theta = \int_{\frac{\pi}{4}}^0 \frac{\sin(\frac{\pi}{2}-2u)+1}{\sin(\frac{\pi}{2}-2u)+\cos(\frac{\pi}{2}-2\pi)+2}(-du)$

$= \int_0^{\frac{\pi}{4}} \frac{\cos 2u+1}{\sin 2u+\cos 2u+2}\, du$

$= \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\, d\theta$

Answers written in the margins will not be marked.

(d) $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta + 9}{\sin 2\theta + \cos 2\theta + 2} d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{8(\sin 2\theta + 1) + 1}{\sin 2\theta + \cos 2\theta + 2} d\theta$

$= 8\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2} d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2} d\theta$

$= 8\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2} d\theta + \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

12. Let $A=\begin{pmatrix}3&1\\0&3\end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Using mathematical induction, prove that $A^n=3^n I+3^{n-1}n\begin{pmatrix}0&1\\0&0\end{pmatrix}$ for all positive integers n.

(4 marks)

(b) Let $B=\begin{pmatrix}5&1\\-4&1\end{pmatrix}$.

(i) Define $P=\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n=3^n I+3^{n-1}n\begin{pmatrix}2&1\\-4&-2\end{pmatrix}$ for any positive integer n.

(iii) Does there exist a positive integer m such that $|A^m-B^m|=4m^2$? Explain your answer.

(8 marks)

(a) Let P(n) be the proposition

For $n=1$, L.H.S. $= A = \begin{pmatrix}3&1\\0&3\end{pmatrix}$

R.H.S. $= 3I + \begin{pmatrix}0&1\\0&0\end{pmatrix}$

$= \begin{pmatrix}3&0\\0&3\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$=\begin{pmatrix}3&1\\0&3\end{pmatrix}$

$=$ L.H.S.

$\therefore$ P(1) is true.

Assume that P(k) is true for some +ve integer k,

i.e. $A^k = 3^k I + 3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}$

For $n=k+1$, R.H.S. $= 3^{k+1}I + 3^k(k+1)\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$=\begin{pmatrix}3^{k+1}&0\\0&3^{k+1}\end{pmatrix}+\begin{pmatrix}0&(3^k)(k+1)\\0&0\end{pmatrix}$

$=\begin{pmatrix}3^{k+1}&(3^k)(k+1)\\0&3^{k+1}\end{pmatrix}$

L.H.S. $= A^{k+1}$

$= A^kA$

$=\left[3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]\left[\begin{pmatrix}3&1\\0&3\end{pmatrix}\right]$

$=\left[\begin{pmatrix}3^k&0\\0&3^k\end{pmatrix}+\begin{pmatrix}0&(3^{k-1})k\\0&0\end{pmatrix}\right]\left[\begin{pmatrix}3&1\\0&3\end{pmatrix}\right]$

Answers written in the margins will not be marked.

$= \begin{pmatrix} 3^k & (3^{k-1})k \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & 3^k + 3^k(3k) \\ 0 & 3^{k+1} \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & 3^k(k+1) \\ 0 & 3^{k+1} \end{pmatrix}$

$=$ R.H.S.

$\therefore P(k+1)$ is true

$\therefore$ By principle of mathematical induction, $P(n)$ is true for all +ve integers n.

(b)(i) $|P| = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$

$\operatorname{adj} P = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}^T = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} = P^{-1}$

$P^{-1}BP = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix} P$

$= \begin{pmatrix} -5 & -1 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

(ii) $P^{-1}BP = A$

$(P^{-1}BP)^n = A^n$

$P^{-1}B^nP = 3^nI + 3^{n-1}n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B^n = P\left[3^nI + 3^{n-1}n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]P^{-1}$

$= 3^nP\,P^{-1} + 3^{n-1}n\,P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}P^{-1}$

$= 3^nI + 3^{n-1}n\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}P^{-1}$

$= 3^nI + 3^{n-1}n\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$

$= 3^nI + 3^{n-1}n\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

END OF PAPER

Answers written in the margins will not be marked.

Comments

The candidate demonstrates comprehensive knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by applying them successfully at a sophisticated level to a wide range of unfamiliar situations in Questions 9, 10, 11 and 12.

He/She is able to communicate and express views and arguments precisely and logically using mathematical language, notations and diagrams, such as in Questions 1, 2, 3, 4, 6, 8, 9, 11 and 12(b).

He/She also provides complex mathematical proofs in a logical, rigorous and concise manner in Questions 6(a), 11(c) and 12(a).

It can be concluded that the candidate has the ability to integrate knowledge and skills from different areas of the curriculum in handling complex tasks using a variety of strategies.